

Single qubit , two qubit gates and no signalling principle

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Abstract

In this work we investigate that whether one can construct single and two qubit gates for arbitrary quantum states from the principle of no signalling. We considered the problem for Pauli gates, Hadamard gate, C-Not gate.

1 Introduction

In quantum information theory, there are many information processing protocols or operations which can not be carried out perfectly for an unknown qubit. This may be probably due to the linear structure or may be due to the unitary evolution in quantum mechanics. Regardless of their origin, these impossible operations are making quantum information processing more restricted than its classical counterpart. On the other hand these restrictions on many quantum information processing tasks are making quantum information more secure. Enlisting of these operations started from the landmark paper of Wootters and Zurek, where 'no-cloning' theorem has been stated [1]. This theorem tells us that one cannot clone a single quantum. Later it was also shown by Pati and Braunstein that we cannot delete either of the two quantum states when we are provided with two identical

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quantum states at our input port [2]. Even after the no-deletion theorem, many other operations like self replication, partial erasure, splitting proved as impossible operations in quantum domain [3,4,5]. These no-go theorems come under the broad heading of General impossible operations [6]. Researches are carried out to see how these no go theorems in quantum information theory are consistent with various principles of quantum mechanics. One of such principle is the principle of no signalling. It tells us that if two distant parties Alice and Bob, share an entangled state, neither Alice nor Bob cannot send signal faster than the speed of light to the other party by doing some local operation on their own subsystems. It had been already seen that if one assumes these impossible operations to be valid physical processes, one can have a super luminal communication between two distant parties sharing an entangled state [7,8,9,10,12,13,14,15,16,17,18]. These results also guarantees impossibility of such operations from the no signalling principle. Here in this work we will address the question that whether one can construct the single qubit and two qubit gates for nonorthogonal states, and we find that it is impossible to do so, as this will violate the principle of no signalling. The entire organization of the work is as follows: In the second section we will discuss the existing proofs of impossibility of various operations from the no signalling principle. In the third section we will consider one qubit gates like Pauli gates, Hadamard gate and will show their impossibility from the principle of no signalling. In the fourth section we will show the same for two qubit gates.

2 Revisiting impossible operations and no signalling principle:

Cloning and no signalling: It is a well known fact that there exists no physical process by which one can achieve the transformation $|\psi_i\rangle \longrightarrow |\psi_i\rangle|\psi_i\rangle$ for a set of non orthogonal states $\{|\psi_i\rangle\}$ [1,11]. One can easily prove that if we assume cloning of an unknown quantum to be a feasible operation, then one can send signals faster than the speed of

light [7]. Let two distant parties Alice and Bob share a singlet state,

$$|X\rangle = \frac{1}{\sqrt{2}}[|\psi\rangle|\bar{\psi}\rangle - |\bar{\psi}\rangle|\psi\rangle] \quad (1)$$

Since the singlet state remains invariant in any arbitrary qubit basis, then after Alice carries out a measurement on her subsystem in any two basis, the resultant reduced density matrix on Bobs side is $\frac{I}{2}$. This clearly indicates that initially under normal scenario, Bob cannot distinguish the statistical mixtures representing his subsystem obtained as a result of measurement carried out by Alice in two different basis. Henceforth it is not possible for Bob to obtain information regarding the basis on which Alice has performed her measurement. However if Bob attaches ancilla to his qubit and perfectly clone his qubit then the entangled state takes the form

$$|X\rangle^C = \frac{1}{\sqrt{2}}[|\psi\rangle|\bar{\psi}\rangle|\bar{\psi}\rangle - |\bar{\psi}\rangle|\psi\rangle|\psi\rangle] \quad (2)$$

Now if Alice performs measurement on her qubit, on two different basis $\{|\psi_1\rangle, |\bar{\psi}_1\rangle\}$ and $\{|\psi_2\rangle, |\bar{\psi}_2\rangle\}$, then the reduced density matrices describing Bobs subsystem are given by

$$\rho_1^C = \frac{1}{2}[|\psi_1\psi_1\rangle\langle\psi_1\psi_1| + |\bar{\psi}_1\bar{\psi}_1\rangle\langle\bar{\psi}_1\bar{\psi}_1|] \quad (3)$$

$$\rho_2^C = \frac{1}{2}[|\psi_2\psi_2\rangle\langle\psi_2\psi_2| + |\bar{\psi}_2\bar{\psi}_2\rangle\langle\bar{\psi}_2\bar{\psi}_2|] \quad (4)$$

Now Bob can easily distinguish the statistical mixture obtained as a result of Alices measurement and subsequently can infer on which basis Alice has performed measurement. This clearly indicates that super luminal signalling has taken place. Hence forth we can conclude that perfect deterministic cloning of an unknown quantum state is not a feasible operation.

General Impossible operations and no signalling: In this subsection we will see that General Impossible Operations[6] which will act on the tensor product of an unknown quantum state and blank state at the input port to produce the original state along with a function of the original state at the output port is not feasible in the quantum world from the no signalling principle [9]. Suppose there is a singlet state consisting of two

particles shared by two distant parties Alice and Bob. The state is given by

$$\begin{aligned} |\chi\rangle_{12} &= \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle) \\ &= \frac{1}{\sqrt{2}}(|\psi\rangle|\bar{\psi}\rangle - |\bar{\psi}\rangle|\psi\rangle) \end{aligned} \quad (5)$$

where $\{|\psi\rangle, |\bar{\psi}\rangle\}$ are mutually orthogonal spin states or in other words they are mutually orthogonal polarizations in case of photon particles. Alice is in possession of the first particle and Bob is in possession of the second particle.

No-signalling principle states that if one distant partner (say, Alice) measures her particle in any one of the two basis namely $\{|0\rangle, |1\rangle\}$ and $\{|\psi\rangle, |\bar{\psi}\rangle\}$, then measurement outcome of the other party (say, Bob) will remain invariant. At this point one might ask an interesting question: Is there any possibility for Bob to know the basis in which Alice measured her qubit, if he applies the operations defined as General Impossible operations[6] on his qubit. Let us consider a situation where Bob is in possession of a hypothetical machine whose action in two different basis $\{|0\rangle, |1\rangle\}$ and $\{|\psi\rangle, |\bar{\psi}\rangle\}$ is defined by the transformation,

$$|i\rangle|\Sigma\rangle \longrightarrow |i\rangle|F(i)\rangle (i = 0, 1) \quad (6)$$

$$|j\rangle|\Sigma\rangle \longrightarrow |j\rangle|F(j)\rangle (j = \psi, \bar{\psi}) \quad (7)$$

where $|\Sigma\rangle$ is the ancilla state attached by Bob. These set of transformations was first introduced by Pati in [6].

After the application of the transformation defined in (6-7) by Bob on his particle, the singlet state takes the form

$$\begin{aligned} |\chi\rangle|\Sigma\rangle &= \frac{1}{\sqrt{2}}(|0\rangle|1\rangle|F(1)\rangle - |1\rangle|0\rangle|F(0)\rangle) \\ &= \frac{1}{\sqrt{2}}(|\psi\rangle|\bar{\psi}\rangle|F(\bar{\psi})\rangle - |\bar{\psi}\rangle|\psi\rangle|F(\psi)\rangle) \end{aligned} \quad (8)$$

Now Alice can measure her particle in two different basis $\{|0\rangle, |1\rangle\}$ and $\{|\psi\rangle, |\bar{\psi}\rangle\}$, then the reduced density matrices describing Bobs subsystem are given by,

$$\rho_1 = \frac{1}{2}[|1\rangle\langle 1| \otimes |F(1)\rangle\langle F(1)| + |0\rangle\langle 0| \otimes |F(0)\rangle\langle F(0)|] \quad (9)$$

$$\rho_2 = \frac{1}{2}[|\bar{\psi}\rangle\langle \bar{\psi}| \otimes |F(\bar{\psi})\rangle\langle F(\bar{\psi})| + |\psi\rangle\langle \psi| \otimes |F(\psi)\rangle\langle F(\psi)|] \quad (10)$$

Since the statistical mixture in (9) and (10) are different, so this would have allow Bob to distinguish the basis in which Alice has performed the measurement and this lead to super luminal signalling. But this is not possible from the principle of no-signalling, so we arrive at a contradiction. Hence, we conclude from the principle of no-signalling that the transformation defined in (6-7) is not possible in the quantum world. Here one can easily see that cloning of a quantum state is a special case of general impossible operations and impossibility of these operations defined in (6-7) from no signalling principle once again proves the impossibility of cloning.

Deletion and no signalling: In reference [8], Pati and Braunstein showed that the deletion of a arbitrary quantum state implies signalling. They have considered a situation where two distant parties Alice and Bob shared two singlet states. The combined state of the system in arbitrary qubit basis $\{|\psi\rangle, |\bar{\psi}\rangle\}$ is given by,

$$\begin{aligned} |\chi\rangle_{12}|\chi\rangle_{34} = & \frac{1}{2} [|\psi\rangle_1|\psi\rangle_2|\bar{\psi}\rangle_3|\bar{\psi}\rangle_4 + |\bar{\psi}\rangle_1|\bar{\psi}\rangle_2|\psi\rangle_3|\psi\rangle_4 \\ & - |\bar{\psi}\rangle_1|\psi\rangle_2|\psi\rangle_3|\bar{\psi}\rangle_4 - |\psi\rangle_1|\bar{\psi}\rangle_2|\bar{\psi}\rangle_3|\psi\rangle_4] \end{aligned} \quad (11)$$

Now if Alice measures her particles in any qubit basis, and if she doesnt convey her measurement result to Bob, then Bobs particles are in completely random mixture i.e $\rho_{24} = \frac{I}{2} \otimes \frac{I}{2}$

But suppose Bob has a quantum deleting machine which can delete arbitrary quantum state. The action of the deleting machine can be described by,

$$\begin{aligned} |\psi\rangle|\psi\rangle|A\rangle & \longrightarrow |\psi\rangle|\Sigma\rangle|A_\psi\rangle \\ |\bar{\psi}\rangle|\bar{\psi}\rangle|A\rangle & \longrightarrow |\bar{\psi}\rangle|\Sigma\rangle|A_{\bar{\psi}}\rangle \\ |\psi\rangle|\bar{\psi}\rangle|A\rangle & \longrightarrow |\phi'\rangle \\ |\bar{\psi}\rangle|\psi\rangle|A\rangle & \longrightarrow |\phi''\rangle \end{aligned} \quad (12)$$

Now if Bob applies the above described deleting machine on his particles, the combined system (11) no longer remains in the previous form. Now if Alice performs measurement on either of two choices of basis states $\{|0\rangle, |1\rangle\}$ and $\{|\psi\rangle, |\bar{\psi}\rangle\}$, then the resultant reduced density matrices describing Bobs subsystem for two different measurement will be

different.

$$\begin{aligned}\rho(0) &= \frac{1}{4}[I_2 \otimes |\Sigma\rangle\langle\Sigma| + \rho'(0) + \rho''(0)] \\ \rho(\theta) &= \frac{1}{4}[I_2 \otimes |\Sigma\rangle\langle\Sigma| + \rho'(\theta) + \rho''(\theta)]\end{aligned}\tag{13}$$

If Alice measures her particle in $\{|0\rangle, |1\rangle\}$ basis, then Bobs particle will be in $\rho(0)$, however if Alice measures her particle in $\{|\psi\rangle, |\bar{\psi}\rangle\}$, then Bobs particle will be represented by $\rho(\theta)$. Thus it is clear that the reduced density matrix describing Bobs subsystem are no longer completely random, but depend upon the choice of basis. Since it is not random Bob can easily distinguish these two density matrices and can infer about the basis on which Alice has performed the measurement. This leads to super luminal signalling. This leads us to contradiction to the initial assumption that perfect deletion is possible.

3 Single Qubit , Two Qubit Gates And No Signalling

Pauli Gates and No signalling: In this subsection we will investigate the question whether one can construct the Pauli gates : X, Y, Z gates, for unknown qubit from the principle of no signalling. X gate: The importance of this gate is immense in quantum information theory. It is also known as a spin flip operator, as it flips a known quantum state into its orthogonal state. However one cannot construct a universal NOT (X-gate) for arbitrary quantum state. However the largest set of states that can be flipped by using single NOT gate is the set lying on a great circle of the Bloch-sphere. In reference [12], authors established this impossibility of construction of universal not gate from the principle of no-signalling. The protocol involved two distant parties sharing an entangled state of the form

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{3}}[|0\rangle_A|0\rangle_B + |1\rangle_A|\psi\rangle_B + |2\rangle_A|\phi\rangle_B]\tag{14}$$

where Alices system is a three dimensional Hilbert space having $\{|0\rangle, |1\rangle, |2\rangle\}$ as basis. Bobs system consists of three states $\{|0\rangle, |\psi\rangle, |\phi\rangle\}$, where

$$\begin{aligned}|\psi\rangle &= a|0\rangle + b|1\rangle \\ |\phi\rangle &= c|0\rangle + d\exp(i\theta)|1\rangle\end{aligned}\tag{15}$$

(where $a^2 + b^2 = c^2 + d^2 = 1; 0 < \theta, \pi; a > 0, c > 0$). Not only that Bob is in possession of hypothetical flipping machine, whose action is defined by

$$\begin{aligned} |0\rangle|M\rangle &\longrightarrow |1\rangle|M_0\rangle \\ |\psi\rangle|M\rangle &\longrightarrow \exp(i\mu)|\bar{\psi}\rangle|M_\psi\rangle \\ |\phi\rangle|M\rangle &\longrightarrow \exp(i\nu)|\bar{\phi}\rangle|M_\phi\rangle \end{aligned} \quad (16)$$

(where μ and ν are some arbitrary phases and $|M\rangle$ is the initial machine state. Initially if we trace out Bobs qubit the reduced density matrix describing Alices subsystem is given by,

$$\rho_A^I = \frac{1}{3}[I + a(|0\rangle\langle 1| + |1\rangle\langle 0|) + c(|0\rangle\langle 2| + |2\rangle\langle 0|) + \langle\psi|\phi\rangle|1\rangle\langle 2| + \langle\phi|\psi\rangle|2\rangle\langle 1|] \quad (17)$$

Now if Bob applies the hypothetical flipping machine (16) on his qubit the entangled state (14) will take a new form and correspondingly the density matrix representation of Alices subsystem will be of the form

$$\begin{aligned} \rho_A^F = \frac{1}{3}[I - a(\exp(-i\mu)\langle M_\psi|M_0\rangle|0\rangle\langle 1| + \exp(i\mu)\langle M_0|M_\psi\rangle|1\rangle\langle 0|) \\ - c(\exp(-i\nu)\langle M_\phi|M_0\rangle|0\rangle\langle 2| + \exp(i\nu)\langle M_0|M_\phi\rangle|2\rangle\langle 0|) + \\ \langle\psi|\phi\rangle \exp(i[\mu - \nu])\langle M_\phi|M_\psi\rangle|1\rangle\langle 2| + \langle\phi|\psi\rangle \exp(i[\nu - \mu])\langle M_\psi|M_\phi\rangle|2\rangle\langle 1|] \end{aligned} \quad (18)$$

Since the flipping operation defined in (16) is a trace preserving quantum operation and there is no classical communication between two distant parties, so from the principle of no signalling one can easily conclude that the two density matrices and will be identical. However a simple calculation reveals that the expressions (17) and (18) are not identical as long as the states are not lying on the same great circle. Henceforth one can conclude that it is impossible to construct a universal NOT gate from the principle of no signalling.

Y gate: This is another single qubit gate, which cannot be constructed for any arbitrary qubit. Here in this subsection we will show that if we assume the construction of this gate for arbitrary qubit, this will lead to the violation of causality. Let us assume that two spatially separated parties Alice and Bob share an entangled state of the form ,

$$|X\rangle_{AB} = \frac{1}{2}[|0\rangle_A|\psi_1\rangle_B + |1\rangle_A|\bar{\psi}_1\rangle_B + |2\rangle_A|\psi_2\rangle_B + |3\rangle_A|\bar{\psi}_2\rangle_B] \quad (19)$$

(where A denotes Alices qubit, while B denotes Bobs qubit). Now if one traces out Bobs qubit, the reduced density matrix describing Alices subsystem will be given by,

$$\rho_A = \frac{1}{4}[I + |0\rangle\langle 2|\langle\psi_2|\psi_1\rangle + |0\rangle\langle 3|\langle\bar{\psi}_2|\psi_1\rangle + |1\rangle\langle 2|\langle\psi_2|\bar{\psi}_1\rangle + |1\rangle\langle 3|\langle\bar{\psi}_2|\bar{\psi}_1\rangle + |2\rangle\langle 0|\langle\psi_1|\psi_2\rangle + |2\rangle\langle 1|\langle\bar{\psi}_1|\psi_2\rangle + |3\rangle\langle 0|\langle\psi_1|\bar{\psi}_2\rangle + |3\rangle\langle 1|\langle\bar{\psi}_1|\bar{\psi}_2\rangle] \quad (20)$$

Let us assume that some how Bob has constructed a hypothetical Y Gate for non orthogonal set of qubits. The action of such a gate is defined by,

$$\begin{aligned} |\psi_i\rangle &\longrightarrow -i|\bar{\psi}_i\rangle \\ |\bar{\psi}_i\rangle &\longrightarrow i|\psi_i\rangle \end{aligned} \quad (21)$$

(where i = 1, 2).

Now if Bob applies the transformation (21) on his qubit, the entangled state reduces to the form

$$|X\rangle_{AB}^F = \frac{1}{2}[-i|0\rangle_A|\bar{\psi}_1\rangle_B + i|1\rangle_A|\psi_1\rangle_B - i|2\rangle_A|\bar{\psi}_2\rangle_B + i|3\rangle_A|\psi_2\rangle_B] \quad (22)$$

As a consequence the reduced density matrix representing Alices subsystem will be of the form

$$\rho_A^Y = \frac{1}{4}[I + |0\rangle\langle 2|\langle\bar{\psi}_2|\bar{\psi}_1\rangle - |0\rangle\langle 3|\langle\psi_2|\bar{\psi}_1\rangle - |1\rangle\langle 2|\langle\bar{\psi}_2|\psi_1\rangle + |1\rangle\langle 3|\langle\psi_2|\psi_1\rangle + |2\rangle\langle 0|\langle\bar{\psi}_1|\psi_2\rangle - |2\rangle\langle 1|\langle\psi_1|\bar{\psi}_2\rangle - |3\rangle\langle 0|\langle\bar{\psi}_1|\psi_2\rangle + |3\rangle\langle 1|\langle\psi_1|\psi_2\rangle] \quad (23)$$

It is clearly evident that the expressions (20) and (23) are not identical for all sets of qubits on the Bloch sphere. However causality demands these expressions to be equal. This is a violation causality. So one can say that it is impossible to construct a universal Y gate.

Z gate: In this subsection we show that it is not possible to construct an universal Z gate by making the construction of such a gate consistent with the principle of no signalling. In other words if we start with a set consisting of non orthogonal quantum states $\{|\psi_i\rangle, |\bar{\psi}_i\rangle\}$ where (i = 1, 2), then from the principle of no signalling one cannot achieve the transformation

$$\begin{aligned} |\psi_i\rangle &\longrightarrow |\psi_i\rangle \\ |\bar{\psi}_i\rangle &\longrightarrow -|\bar{\psi}_i\rangle \end{aligned} \quad (24)$$

In order to have a proof of the above statement, quite likely to other proofs we consider a situation where two distant partners are sharing an entangled state of the form

$$|X\rangle_{AB} = \frac{1}{2} [|0\rangle_A |\psi_1\rangle_B + |1\rangle_A |\bar{\psi}_1\rangle_B + |2\rangle_A |\psi_2\rangle_B + |3\rangle_A |\bar{\psi}_2\rangle_B] \quad (25)$$

One can easily obtain the reduced density matrix of Alices system in order to have an idea of her subsystem. The reduced density matrix describing Alice subsystem is given by

$$\begin{aligned} \rho_A = \frac{1}{4} [& |0\rangle\langle 2| \langle \psi_2 | \psi_1 \rangle + |0\rangle\langle 3| \langle \bar{\psi}_2 | \psi_1 \rangle + |1\rangle\langle 2| \langle \psi_2 | \bar{\psi}_1 \rangle + |1\rangle\langle 3| \langle \bar{\psi}_2 | \bar{\psi}_1 \rangle \\ & + |2\rangle\langle 0| \langle \psi_1 | \psi_2 \rangle + |2\rangle\langle 1| \langle \bar{\psi}_1 | \psi_2 \rangle + |3\rangle\langle 0| \langle \psi_1 | \bar{\psi}_2 \rangle + |3\rangle\langle 1| \langle \bar{\psi}_1 | \bar{\psi}_2 \rangle] \end{aligned} \quad (26)$$

The no signalling principle demands that one cannot send information with a speed faster than the speed of light. In other words if one of the two distant partners carries out local on his qubit, it will not change the reduced density matrix of other party instantaneously. However we find here that if Bob applies this gate defined by (24) on his qubit, the reduced density matrix describing Alices subsystem will be different from what it was initially. The reduced density matrix describing Alices subsystem after Bobs application of hypothetical Z gate on his qubit, will be of the form

$$\begin{aligned} \rho_A^F = \frac{1}{4} [& |0\rangle\langle 2| \langle \psi_2 | \psi_1 \rangle - |0\rangle\langle 3| \langle \bar{\psi}_2 | \psi_1 \rangle - |1\rangle\langle 2| \langle \psi_2 | \bar{\psi}_1 \rangle + |1\rangle\langle 3| \langle \bar{\psi}_2 | \bar{\psi}_1 \rangle \\ & + |2\rangle\langle 0| \langle \psi_1 | \psi_2 \rangle - |2\rangle\langle 1| \langle \bar{\psi}_1 | \psi_2 \rangle - |3\rangle\langle 0| \langle \psi_1 | \bar{\psi}_2 \rangle + |3\rangle\langle 1| \langle \bar{\psi}_1 | \bar{\psi}_2 \rangle] \end{aligned} \quad (27)$$

It is clearly evident that equations (26) and (27) are not identical. This clearly indicates that super luminal signalling has taken place, which is an impossible phenomenon in principle. So we arrive at a contradiction and conclude that, one cannot design universal Z gate as it will violate the principle of no signalling.

Hadamard gate and No signalling principle: This is yet another gate which has got immense application in quantum information theory. The interesting question is that can we design a universal Hadamard gate. What does no signalling principle tells us? The answer to this question is no. In references [9,19] authors showed that construction of universal Hadamard gate will violate no signalling principle. In this section we put forward a proof used in those references.

Now, we define the Hadamard transformation for arbitrary qubit in the following way:

$$\begin{aligned} |\psi_i\rangle|M\rangle &\longrightarrow \frac{1}{\sqrt{2}}(|\psi_i\rangle + e^{i\phi_i}|\bar{\psi}_i\rangle)|H_{\psi_i}\rangle \\ |\bar{\psi}_i\rangle|M\rangle &\longrightarrow \frac{1}{\sqrt{2}}(|\psi_i\rangle - e^{i\phi_i}|\bar{\psi}_i\rangle)|H_{\bar{\psi}_i}\rangle \end{aligned} \quad (28)$$

(i=1,2), where $\langle\psi_i|\bar{\psi}_i\rangle = 0$.

The entangled state shared between two distant partners is given by

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}[|0\rangle|\psi_1\rangle + |1\rangle|\psi_2\rangle]|M\rangle \quad (29)$$

Before and after the application of Hadamard transformation on Bobs qubit, the reduced density matrices describing the Alices subsystem are given by,

$$\rho_A = \frac{1}{2}[|0\rangle\langle 0| + |1\rangle\langle 1| + |0\rangle\langle 1|(\langle\psi_2|\psi_1\rangle) + |1\rangle\langle 0|(\langle\psi_1|\psi_2\rangle)] \quad (30)$$

and

$$\begin{aligned} \rho_A^H &= \frac{1}{4}[|0\rangle\langle 0|(2) + |1\rangle\langle 1|(2) \\ &+ |0\rangle\langle 1|(\langle\psi_2|\psi_1\rangle + \langle\bar{\psi}_2|\psi_1\rangle + \langle\psi_2|\bar{\psi}_1\rangle + \langle\bar{\psi}_2|\bar{\psi}_1\rangle)(\langle H_{\psi_2}|H_{\psi_1}\rangle) \\ &+ |1\rangle\langle 0|(\langle\psi_1|\psi_2\rangle + \langle\bar{\psi}_1|\psi_2\rangle + \langle\psi_1|\bar{\psi}_2\rangle + \langle\bar{\psi}_1|\bar{\psi}_2\rangle)(\langle H_{\psi_1}|H_{\psi_2}\rangle)] \end{aligned} \quad (31)$$

It is clear from equations (31) and (32) that the reduced density matrices are different. This implies that by designing the perfect Hadamard gate, one can send information faster than light, which is impossible. Hence perfect construction of universal Hadamard gate is not possible.

4 Two Qubit Gates and No signalling principle:

C-Not Gate and No signalling: In this section we will show that it is impossible to construct C-Not gates for a set of non orthogonal qubits from the no signalling principle. Controlled -Not gate is a two qubit gate which acts as a Not gate to the second qubit (target qubit), when the first qubit (control qubit) is set to lie in the computational basis

$\{|0\rangle, |1\rangle\}$. The action of this two qubit gate in the computational basis $\{|0\rangle, |1\rangle\}$, is given by,

$$\begin{aligned}
|0\rangle|0\rangle &\longrightarrow |0\rangle|0\rangle \\
|0\rangle|1\rangle &\longrightarrow |0\rangle|1\rangle \\
|1\rangle|0\rangle &\longrightarrow |1\rangle|1\rangle \\
|1\rangle|1\rangle &\longrightarrow |1\rangle|0\rangle
\end{aligned} \tag{32}$$

At this point one may ask an interesting question that if we are provided with a set consisting of non orthogonal quantum states $|\psi_i\rangle$ is it possible for us to construct such a gate. Let us assume that construction of such a gates for non orthogonal states is possible. The action of such a gate is described by,

$$\begin{aligned}
|\psi_i\rangle|\psi_i\rangle &\longrightarrow |\psi_i\rangle|\psi_i\rangle \\
|\psi_i\rangle|\bar{\psi}_i\rangle &\longrightarrow |\psi_i\rangle|\bar{\psi}_i\rangle \\
|\bar{\psi}_i\rangle|\psi_i\rangle &\longrightarrow |\bar{\psi}_i\rangle|\bar{\psi}_i\rangle \\
|\bar{\psi}_i\rangle|\bar{\psi}_i\rangle &\longrightarrow |\bar{\psi}_i\rangle|\psi_i\rangle
\end{aligned} \tag{33}$$

Let us consider the situation where two distant parties Alice and Bob share an entangled state of the form,

$$|X\rangle = \frac{1}{2}[|0\rangle_A(|\bar{\psi}_1\rangle|\psi_1\rangle)_B + |1\rangle_A(|\bar{\psi}_1\rangle|\bar{\psi}_1\rangle)_B + |2\rangle_A(|\bar{\psi}_2\rangle|\psi_2\rangle)_B + |3\rangle_A(|\bar{\psi}_2\rangle|\bar{\psi}_2\rangle)_B] \tag{34}$$

where $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$ are the basis vectors of the Hilbert space describing Alices subsystem. Now after tracing out Bobs qubit the reduced density matrix describing Alices subsystem is given by

$$\begin{aligned}
\rho_A = \frac{1}{4}[I &+ |2\rangle\langle 0|\{\langle\bar{\psi}_1|\bar{\psi}_2\rangle\langle\psi_1|\psi_2\rangle\} + |3\rangle\langle 0|\{\langle\bar{\psi}_1|\bar{\psi}_2\rangle\langle\psi_1|\bar{\psi}_2\rangle\} \\
&+ |2\rangle\langle 1|\{\langle\bar{\psi}_1|\bar{\psi}_2\rangle\langle\bar{\psi}_1|\psi_2\rangle\} + |3\rangle\langle 1|\{\langle\bar{\psi}_1|\bar{\psi}_2\rangle\langle\bar{\psi}_1|\bar{\psi}_2\rangle\} \\
&+ |0\rangle\langle 2|\{\langle\bar{\psi}_2|\bar{\psi}_1\rangle\langle\psi_2|\psi_1\rangle\} + |1\rangle\langle 2|\{\langle\bar{\psi}_2|\bar{\psi}_1\rangle\langle\psi_2|\bar{\psi}_1\rangle\} \\
&+ |0\rangle\langle 3|\{\langle\bar{\psi}_2|\bar{\psi}_1\rangle\langle\bar{\psi}_2|\psi_1\rangle\} + |1\rangle\langle 3|\{\langle\bar{\psi}_2|\bar{\psi}_1\rangle\langle\bar{\psi}_2|\bar{\psi}_1\rangle\}]
\end{aligned} \tag{35}$$

Now if Bob applies the C-Not gate ,defined by equation (34), on his qubit, then the initially shared entangled state takes the form

$$|X\rangle^{C-Not} = \frac{1}{2}[|0\rangle_A(|\bar{\psi}_1\rangle|\bar{\psi}_1\rangle)_B + |1\rangle_A(|\bar{\psi}_1\rangle|\psi_1\rangle)_B + |2\rangle_A(|\bar{\psi}_2\rangle|\bar{\psi}_2\rangle)_B + |3\rangle_A(|\bar{\psi}_2\rangle|\psi_2\rangle)_B] \tag{36}$$

Henceforth the reduced density matrix describing Bobs subsystem is given by,

$$\begin{aligned}\rho_A^{C-Not} = \frac{1}{4} & [I + |2\rangle\langle 0| \{ \langle \bar{\psi}_1 | \bar{\psi}_2 \rangle \langle \bar{\psi}_1 | \bar{\psi}_2 \rangle \} + |3\rangle\langle 0| \{ \langle \bar{\psi}_1 | \bar{\psi}_2 \rangle \langle \bar{\psi}_1 | \psi_2 \rangle \} \\ & + |2\rangle\langle 1| \{ \langle \bar{\psi}_1 | \bar{\psi}_2 \rangle \langle \psi_1 | \bar{\psi}_2 \rangle \} + |3\rangle\langle 1| \{ \langle \bar{\psi}_1 | \bar{\psi}_2 \rangle \langle \psi_1 | \psi_2 \rangle \} \\ & + |0\rangle\langle 2| \{ \langle \bar{\psi}_2 | \bar{\psi}_1 \rangle \langle \bar{\psi}_2 | \bar{\psi}_1 \rangle \} + |1\rangle\langle 2| \{ \langle \bar{\psi}_2 | \bar{\psi}_1 \rangle \langle \bar{\psi}_2 | \psi_1 \rangle \} \\ & + |0\rangle\langle 3| \{ \langle \bar{\psi}_2 | \bar{\psi}_1 \rangle \langle \psi_2 | \bar{\psi}_1 \rangle \} + |1\rangle\langle 3| \{ \langle \bar{\psi}_2 | \bar{\psi}_1 \rangle \langle \psi_2 | \psi_1 \rangle \}] \quad (37)\end{aligned}$$

Now it is clearly evident that equations (36) and (38) are not identical. This indicates that the action of C-Not gate on Bobs qubit caused the change in the density matrix describing Alices subsystem. In other words we can say that local action performed by Bob on his qubit allowed super luminal signalling to take place. But in reality, this is not possible. This leads us into a contradiction and henceforth we conclude that C-Not gate for non orthogonal set of qubits cannot exist in reality.

(38)

5 Conclusion:

Here in this work we presented a systematic overview of the existing impossible operations in quantum domain and their relationship with no signalling principle. In this work we not only demonstrate the existing impossibility proofs of various physical operations but also showed the impossibility of construction of gates like pauli gates and C-Not gate for arbitrary qubits from the no signalling principle. As these gates are the building block for universal quantum gates, one may look out for the answer that whether the construction of universal quantum gates for arbitrary qubit is possible from no signalling principle or not.

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